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LETTER TO THE EDITOR

Exact solution of an N-body problem in one dimension

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Abstract. A complete energy spectrum is obtained for the quantum mechanical problem of N one-dimensional equal mass particles interacting via potential

$$V(x_1, x_2, \ldots, x_N) = g \sum_{i<j}^{N} \frac{1}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}}.$$

Furthermore, it is shown that the scattering configuration, characterized by initial momenta $p_i (i = 1, 2, \ldots, N)$, goes over into a final configuration characterized uniquely by the final momenta p_i' with $p_i' = p_{N+1-i}$.

In recent years, the Calogero–Sutherland (CS) type of N-body problems in one dimension has received considerable attention in the literature [1–4]. It is believed that the CS model with inverse square interaction provides an example of an ideal gas in one dimension with fractional statistics [5]. Moreover, these models are related to (1 + 1)-dimensional conformal field theory, random matrices, as well as a host of other things [6]. Inspired by these successes, it is of considerable interest to discover new exactly solvable N-body problems.

The purpose of this letter is to present one such example. In particular, we show that the N-body problem with equal mass in one dimension, characterized by ($\hbar = 2m = 1$, $g > -1/2$, $\alpha > 0$)

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N} \frac{g}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}} \tag{1}$$

is exactly solvable. The interesting point about this model is that, unlike most other exactly solvable models, it has both bound-state and scattering solutions. In particular, we show that the complete bound-state spectrum (in the centre-of-mass frame) is given by the formula

$$E_{n+k} = -\frac{\alpha^2}{4N(n + k + b + \frac{1}{2})^2} \qquad n, k = 0, 1, 2 \tag{2}$$

where

$$b = +\frac{N(N-1)}{2}a + \frac{N(N+1)}{4} - \frac{3}{2} \qquad a = \frac{1}{2}\sqrt{1 + 2g}. \tag{3}$$

For positive energy one has only scattering states. We show that a scattering configuration, characterized by initial momenta p_i ($i = 1, 2, \ldots, N$), goes over into a final configuration characterized uniquely by the final momenta p_i' with

$$p_i' = p_{N+1-i}. \tag{4}$$

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However, unlike the pure inverse square scattering case ($\alpha = 0$), in our case the phase shift is energy-dependent. Thus, as in other integrable cases, the scattering problem reduces to a sequence of two-body processes.

Finally, following Sutherland [3], I also solve a slightly different variant of the Hamiltonian (1), with $-\alpha/\sqrt{\sum_{i<j}(x_i - x_j)^2}$ being replaced by a potential $-\alpha/\sqrt{\sum_i x_i^2}$, and obtain exact expressions for the ground-state energy eigenvalues and eigenfunctions.

Consider the Hamiltonian as given by equation (1). We need to solve the eigenvalue equation

$$H\psi = E\psi \tag{5}$$

where ψ is a translation invariant eigenfunction. Note that our Hamiltonian is very similar to the classic Calogero Hamiltonian (see equation (2.1) of his paper [2]) except that, whereas he has a pairwise quadratic potential, we have a 'N-body' potential $-\alpha/\sqrt{\sum_{i<j}(x_i - x_j)^2}$. However, we shall see that many of the key steps are very similar in the two cases and hence we avoid giving most steps which are already contained there [2]. Without any loss of generality, as in [2] we also restrict our attention to the sector of the configuration space corresponding to a definite ordering of particles, say

$$x_i \geqslant x_{i+1} \qquad i = 1, 2, \ldots, N-1. \tag{6}$$

From [2] it is clear that the normalizable solutions of equation (5) (with H being given by equation (1)) can be cast in the form

$$\psi(x) = Z^{a+1/2}\phi(r)P_k(x) \tag{7}$$

where a is defined in equation (3), while Z and r are given by

$$Z = \Pi_{i<j}^{N}(x_i - x_j) \qquad r^2 = \frac{1}{N}\sum_{i<j}^{N}(x_i - x_j)^2 \tag{8}$$

and $P_k(x)$ is a homogeneous polynomial of degree k in the particle coordinates and satisfies the generalized Laplace equation, i.e.

$$\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\left(a + \frac{1}{2}\right)\sum_{i<j}^{N}\frac{1}{(x_i - x_j)}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right]P_k(x) = 0. \tag{9}$$

As discussed in detail in [2], the polynomials $P_k(x)$ are completely symmetrical under the exchange of any two coordinates. On inserting the ansatz (7) into the Schrödinger equation (5) (with H given by equation (1)) and using equation (9) and following the procedure of [2], we find that $\phi(r)$ satisfies the equation

$$-\left[\phi''(r) + \{2k + 2b + 1\}\frac{1}{r}\phi'(r)\right] - (\frac{\alpha}{\sqrt{N}r} + E)\phi(r) = 0 \tag{10}$$

where prime denotes differentiation with respect to the argument. The normalizable solutions of this equation are

$$\phi_{n,k}(r) = \exp\left(-\sqrt{|E|}r\right)L_n^{2b+2k}\left(2\sqrt{|E|}r\right) \tag{11}$$

while the corresponding energies are as given by equation (2). Here $L_n^{\alpha}(r)$ is a Laguerre polynomial. Notice that in expression (2) for the energy, n and k always come in the combination $n+k$ (unlike the Calogero case [2] where it comes in the combination $2n+k$).

In the special case of $N = 3$, we can check our expressions for E_n and $\psi_{n,k}$ with the exact expressions obtained by an entirely different method (see equations (40)–(43) of [7]).

On comparing the two we find (note that the coupling constant in [7] is $\sqrt{3}\alpha$ rather than α) that the two expressions agree provided $k = 3l$ and $P_k(x) \propto r^{3l} C_l^{a+1/2}(\cos 3\phi)$ where C_i^a is a Gegenbauer polynomial.

Let us now consider the positive energy spectrum of the Hamiltonian (1). It is, of course, a purely continuous spectrum. Following the treatment given above and as in [2, section 4], it is clear that the complete set of stationary eigenfunctions of the problem (in the centre-of-mass frame) is

$$\psi_{pk} = Z^{a+1/2}\phi_p(r)P_k(x) \qquad k = 0, 1, 2, \ldots \qquad p \geqslant 0 \tag{12}$$

where p is connected to the energy eigenvalue by $E = p^2 \geqslant 0$ (note that we have chosen $\hbar = 2m = 1$) while $\phi_p(r)$ satisfies equation (10). It is easily shown that for $E \geqslant 0$, the solution of equation (10) is given by

$$\phi_p(r) = \mathrm{e}^{\mathrm{i}pr} F\left(k + b + \frac{1}{2} - \frac{\mathrm{i}\alpha}{2p\sqrt{N}}, 2k + 2b + 1; -2\mathrm{i}pr\right). \tag{13}$$

One can now run through the arguments of [2, section 4] and show that if the stationary eigenfunction describing, in the centre-of-mass frame, the scattering situation is characterized by the form

$$\psi_{in} \sim C \exp\left(\mathrm{i}\sum_{i=1}^{N} p_i x_i\right) \tag{14}$$

with (note $x_i \geqslant x_{i+1}$, $i = 1, 2, \ldots, N-1$)

$$p_i \leqslant p_{i+1} \qquad p^2 = \sum_{i=1}^{N} p_i^2 \qquad \sum_{i=1}^{N} p_i = 0 \tag{15}$$

then ψ_{out} is given by

$$\psi_{\mathrm{out}} \sim C\mathrm{e}^{2\mathrm{i}\eta_p - ib\pi} \exp\left(\mathrm{i}\sum_{i=1}^{N} p_{N+1-i} x_i\right) \tag{16}$$

where

$$\mathrm{e}^{2\mathrm{i}\eta_p} = \frac{\Gamma(k + b + \frac{1}{2} - \mathrm{i}\alpha/2p\sqrt{N})}{\Gamma(k + b + \frac{1}{2} + \mathrm{i}\alpha/2p\sqrt{N})}. \tag{17}$$

Thus we have the remarkable result that, even in the presence of the potential $-\alpha/\sqrt{\sum_{i<j}(x_i - x_j)^2}$, the N-particle scattering problem reduces to a sequence of two-body processes as characterized by equation (4) but now one has an energy-dependent phase shift. Note that all the results about scattering are also valid in the case when α is negative but now the spectrum is purely continuous and there are no bound states.

Finally, let us discuss the 'Sutherland variant' [3] of the Hamiltonian (1). Consider

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{g}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_i x_i^2}} \tag{18}$$

i.e. the N-body potential is now slightly different. Note that for the Calogero case, Sutherland [3] was able to obtain an exact expression for the ground-state energy and eigenfunction ψ, and find a remarkable connection of ψ^2 with the joint probability density function for the eigenvalues of matrices from a Gaussian ensemble in the case when $\beta = 2\lambda = 1$, 2 or 4. Using these connections he was able to compute [3] the one-particle density and the pair correlation function.

Following Sutherland, let us consider the Schrödinger equation $H\psi = E\psi$ with H as given by equation (18). Furthermore, let us write the wavefunction ψ as $\psi = \phi\Phi$ with

$$\phi = \Pi_{i<j}|x_i - x_j|^{\lambda} \qquad \lambda = \tfrac{1}{2} + a. \tag{19}$$

On using this ansatz in the Schrödinger equation with H as given by equation (18) we find that Φ must satisfy

$$-\sum_{i=1}^{N} \frac{\partial^2 \Phi}{\partial x_i^2} - 2\lambda \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Phi - \frac{e^2}{\sqrt{\sum_i x_i^2}} \Phi = E\Phi. \tag{20}$$

It is easily verified that

$$\Phi = \exp\left(-\sqrt{|E|} \sqrt{\sum_i x_i^2} \right) \tag{21}$$

is a solution to equation (20) with the energy

$$E = -\frac{e^4}{[(N-1)(1+\lambda N)]^2}. \tag{22}$$

Clearly, for each ordering of particles, ψ is nodeless and hence it is the solution for the ground state. If we rewrite ψ in terms of the variables

$$y_i = \frac{\sqrt{|E|}}{\sqrt{\lambda}} x_i \tag{23}$$

then one finds that

$$\psi^2 = C \exp\left(-\beta \sqrt{\sum_i y_i^2} \right) \Pi_{i<j}|y_i - y_j|^{\beta} \tag{24}$$

where C is the normalization constant. Following the original Sutherland case [3], where ψ^2 was identical to the joint probability density function, it would indeed be remarkable if our ψ^2 as given by equation (24), for at least $\beta = 1, 2, 4$, can be mapped onto some known solvable problem and, using these results, if one could obtain the one-particle density and the pair correlation function for our case.

This work raises several issues which need to be looked into. I hope to address some of these issues in the near future.

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2191, 2197

[2] Calogero F 1971 *J. Math. Phys.* **12** 419

[3] Sutherland B 1971 *J. Math. Phys.* **12** 246

[4] For excellent review of this field see Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313; 1983 *Phys. Rep.* **94** 313

[5] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
Wu Y-S 1994 *Phys. Rev. Lett.* **73** 922
Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **73** 3331

[6] See for example the flow chart with various connections in Simons B D, Lee P A and Altshuler B L 1994 *Phys. Rev. Lett.* **72** 64

[7] Khare A and Bhaduri R K 1994 *J. Phys. A: Math. Gen.* **27** 2213

[8] See for example, Mehta M L 1990 *Random Matrices* revised edn (New York: Academic)